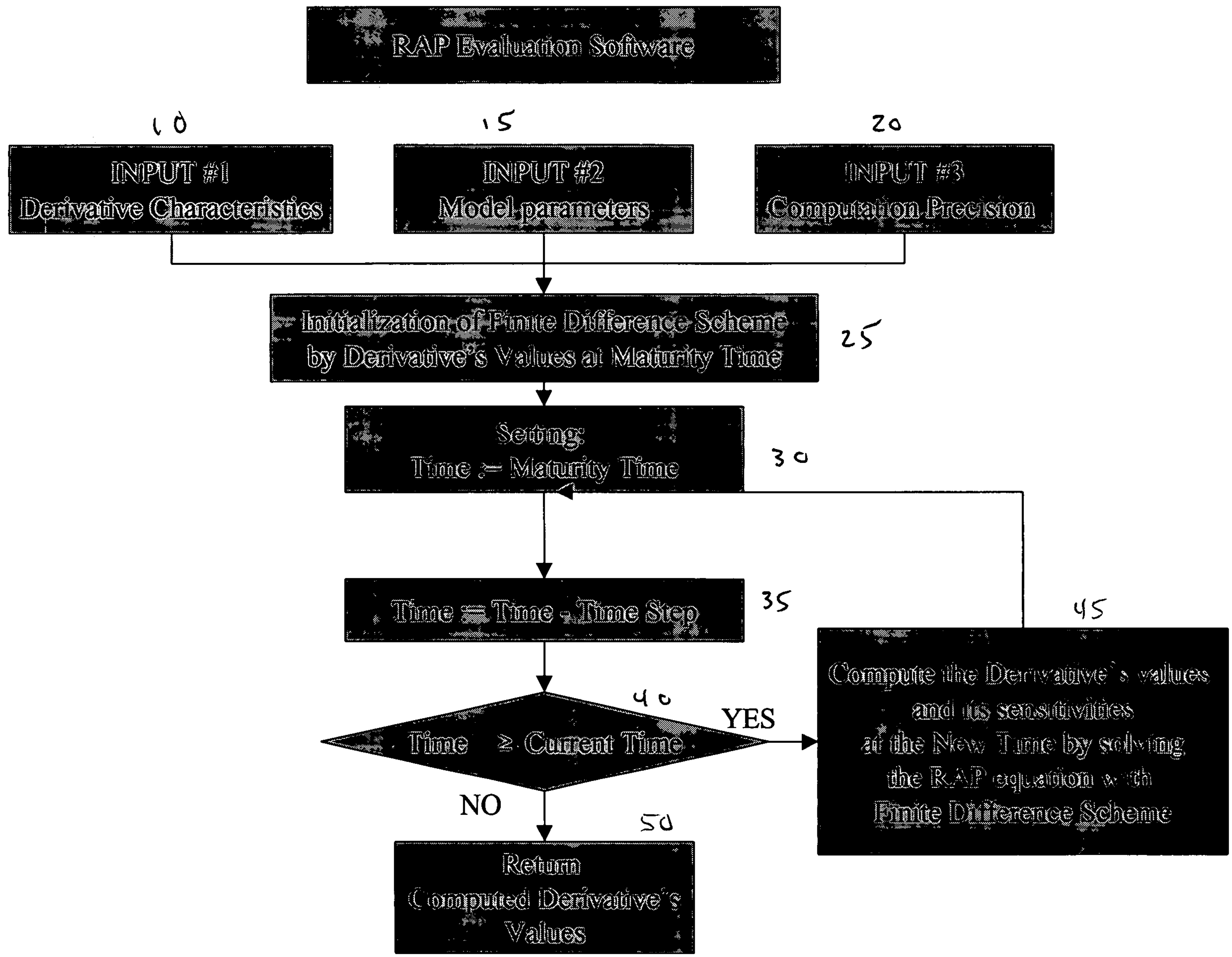
Figure 1



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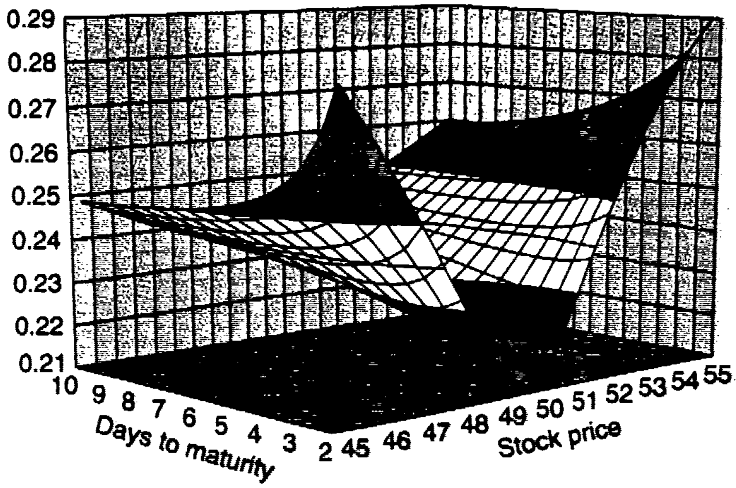
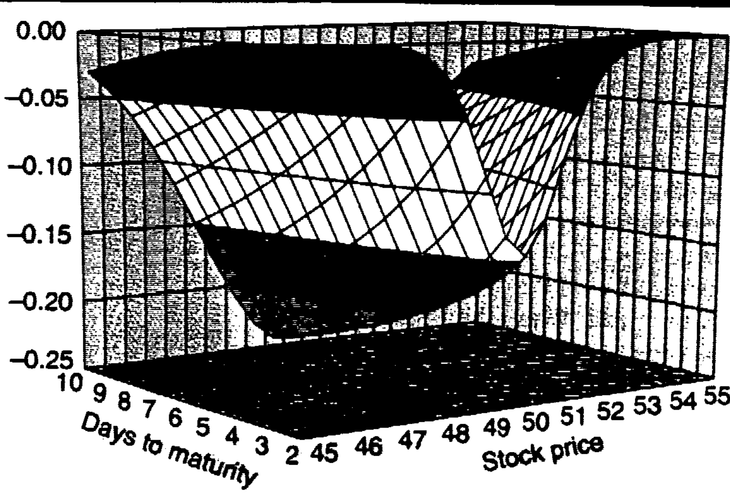
1. Black-Scholes implied volatility

Figure 2



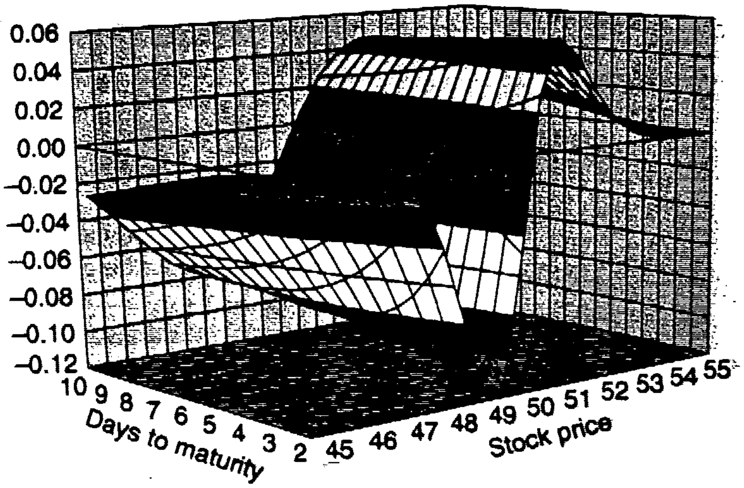
2. Risk-adjusted call price minus Black-Scholes call price

Figure 3



3. Risk-adjusted delta minus Black-Scholes delta

Figure 4



4. Risk-adjusted gamma minus Black-Scholes gamma

Figure 5

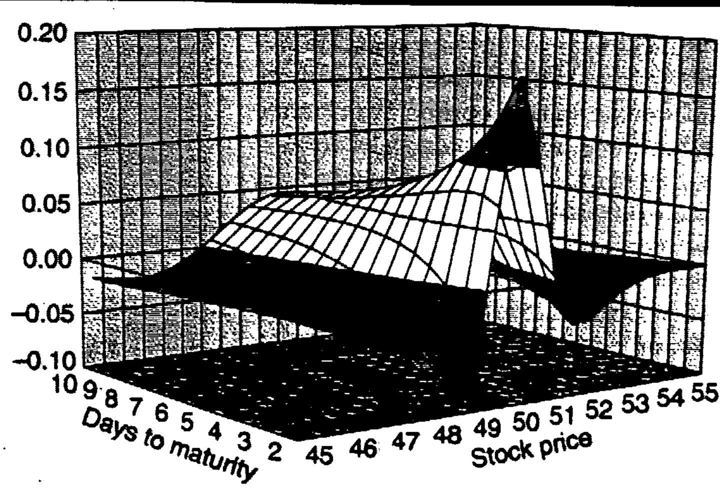


Figure 6

```
#include <math.h>
#include <stdlib.h>
#include <iostream.h>

#define  MAX_TIME_STEPS  200
#define  MAX_PRICE_STEPS 200
#define  pi 3.1415962
#define  PUT 'P'
#define  CALL 'C'
#define  MAX_GAMMA 128.0

#define max(a,b)    (((a) > (b)) ? (a) : (b))
#define min(a,b)    (((a) < (b)) ? (a) : (b))

double O[MAX_TIME_STEPS][MAX_PRICE_STEPS]; // array of the option price
double Delta[MAX_TIME_STEPS][MAX_PRICE_STEPS]; // array of deltas
double Gamma[MAX_TIME_STEPS][MAX_PRICE_STEPS]; // array of gammas
double Theta[MAX_TIME_STEPS][MAX_PRICE_STEPS]; // array of thetas
double Vega[MAX_TIME_STEPS][MAX_PRICE_STEPS]; // array of vegas
double M[MAX_TIME_STEPS][MAX_PRICE_STEPS]; // array of the volatility adjust
ments
int S0;

void RAPoptionEuropean(// option's characteristics:
               double T,   // T is time to maturity in years
               double K,   // K is the strike price in dollars
               double S,   // S is the current underlying price in dollars
               char Y,     // Y is the option type: Put or Call
               // model parameters:
               double sig, // sig is the implied underlying volatility
               double mu,  // expected trend rate
               double R,   // R is the variance risk premium coeff.
               double C,   // c is the transaction costs/slippage coeff.
               double v,   // v is the implied volatility risk premium coeff

               double g,   // gap risk coeff.
               double r,   // risk free rate
               double rb,  // borrowing fees rate
               double rd,  // dividend yield rate
               // computational parameters:
               int nt,  // number of time steps
               int ns,  // number of price steps
               double ls,  // lower bound on underlying price S
               double us  // upper bound on underlying price S
               ){

  //  DO: initialize the computational variables
  double m1 = pow( 324.0 * R * C * C / pi, 1.0/3.0 ); // m is the risk param
eter
  double dt = T / nt;                // time step
  double ds = ( us - ls ) / ns;      // the price step size
  double s[MAX_PRICE_STEPS];         // array of the underlying stock prices
  double s2[MAX_PRICE_STEPS];        // array of squares of the underlying stock
 prices
  double t[MAX_TIME_STEPS];          // array of the time steps
  double o[MAX_PRICE_STEPS];         // array of the option price
  double delta[MAX_PRICE_STEPS]; // array of deltas
  double gamma[MAX_PRICE_STEPS]; // array of deltas
  double theta[MAX_PRICE_STEPS]; // array of deltas
```

Figure 6 (cont.)

```
double vega[MAX_PRICE_STEPS];   // array of vegas
double m[MAX_PRICE_STEPS];      // array of the volatility adjustments
double o_old[MAX_PRICE_STEPS];      // array of old option prices

int it;  // an auxiliary index variable
int is;  // an auxiliary index variable
double ds2 = ds * ds;
double a[MAX_PRICE_STEPS]; // array of matrix coefficients
double b[MAX_PRICE_STEPS]; // array of matrix coefficients
double c[MAX_PRICE_STEPS]; // array of matrix coefficients
double d[MAX_PRICE_STEPS]; // array of matrix coefficients
double aux1, aux2, aux3, aux4, aux5, aux6;

//  DO: Make sure that the coefficients are within their meaningfull range
if( nt > MAX_TIME_STEPS || nt < 1 ) nt = MAX_TIME_STEPS;
if( ns > MAX_PRICE_STEPS || ns < 1 ) ns = MAX_PRICE_STEPS;
if( sig < 0.0 ) sig = - sig;  // volatility cannot be negative
if( R < 0.0 ) R = -R;         // variance risk premium cannot be negative
if( g < 0.0 ) g = -g;         // gap risk premium cannot be negative
if( rb < 0.0 ) rb = -rb;      // borrowing fees cannot be nagative
if( ls < 0.0 ) ls = 0.0;        // underlying price range cannot be negati
ve
if( us < 0.0 ) us = -us;        // underlying price range cannot be negati
ve
v = 0.0;  // set the volatility risk premium to zero in this demonstration
procedure

//  DO: Set the borrowing rate and dividend rate for puts to zero
if( Y == PUT ){
  rb = 0.0;
}

//  DO: Initialize the arrays of the time
for( it = 0; it <= nt; it++ ){
  t[it] = it * dt; // the maturity time T is t[nt], the current time 0 is
t[0]
}

//  DO: Initialize the arrays of the price
is = 0;
s[0] = S - is * ds;
while( s[0] >= ls ){
  s[0] = s[0] - ds;
  is++;
}
S0 = is - 1;
s[0] = s[0] + ds;
for( is = 1; is <= ns; is++ ){
  s[is] = s[is-1] + ds;
}
for( is = 0; is <= ns; is++ ){
  s2[is] = s[is] * s[is];  // array of the price squares
}

//  DO: initialize the finite difference scheme by the derivative's values
//      at maturity time by the payoff function
if( Y == CALL ){
  for( is = 0; is <= ns; is++ ){
    o[is] = max( 0.0, s[is] - K );  // call option payodeltasff
```

Figure 6 (cont.)

```
    if( s[is] < K ){
  delta[is] = 0.0;  // delta zero below the strike
  gamma[is] = 0.0;  // gamma zero
    } else if( s[is] == K ){
  delta[is] = 0.5;  // at the strike delta 0.5 for numerical stability
  gamma[is] = MAX_GAMMA;  // gamma is the maximum allowed by the market
                          // which is inverse of the minimum tick size
    } else if( s[is] > K ){
  delta[is] = 1.0;  // delta one above the strike
  gamma[is] = 0.0;  // gamma zero
    }
    theta[is] = 0.0;  // no time decay left
    vega[is] = 0.0;   // no volatility sensitivity
  }
} else if( Y == PUT ){
  for( is = 0; is <= ns; is++ ){
    o[is] = max( 0.0, K - s[is] );  // put option payoff
    if( s[is] < K ){
  delta[is] = -1.0;  // delta zero below the strike
  gamma[is] = 0.0;  // gamma zero
    } else if( s[is] == K ){
  delta[is] = 0.5;  // at the strike delta 0.5 for numerical stability
  gamma[is] = MAX_GAMMA;  // gamma is the maximum allowed by the market
                          // which is inverse of the minimum tick size
    } else if( s[is] > K ){
  delta[is] = 0.0;  // delta one above the strike
  gamma[is] = 0.0;  // gamma zero
    }
    theta[is] = 0.0;  // no time decay left
    vega[is] = 0.0;   // no volatility sensitivity
  }
}

for( is = 0; is <= ns; is++ ){
  if( gamma[is] > 0.0 ){
    m[is] = max( 0.0, 1.0 + g - min( m1 * pow( gamma[is], 1.0/3.0 ),
                    2.0 * R * delta[is] * delta[is] / gamma[is] ) );
  } else {
    m[is] = 1.0 + g;
  }
}

for( is = 0; is <= ns; is++ ){
  O[nt][is]     = o[is];
  Delta[nt][is] = delta[is];
  Gamma[nt][is] = gamma[is];
  Theta[nt][is] = theta[is];
  Vega[nt][is]  = vega[is];
}

//  DO: remember the last computed values
for( is = 0; is <= ns; is++ ){
  o_old[is]     = o[is];
 }

//  DO: Setting: starting time is maturity time
it = nt;

// DO: set the auxiliary variables for faster computational speed
```

Figure 6(cont).

```
  aux1 = 0.25 * sig * sig * dt / ( ds * ds );
  aux2 = 0.25 * ( r - rb - rd ) * dt / ds;
  aux3 = 0.5 * r * dt;

  //  DO: in the loop compute all the optionis prices
  while( it >= 0 ){ // INV: new time >= current time

    it--;  // Meaning: Time := Time - Time Step

    // DO: compute the solution on the new time level by solving
    //     the RAP equation with the semi-implicit finite difference
    //     Crank-Nicolson scheme method

    // DO: Initialize the triangular matrix coefficients and the right hand
side
    for( is = 1; is < ns; is++ ){
      // a,b,c are on the diagonal
      aux4 = aux1 * s2[is] * m[is];
      aux5 = aux2 * s[is];

      a[is] = aux4 - aux5;
      b[is] = -1.0 - 4 * aux4 - aux3;
      c[is] = aux4 + aux5;
      // d is the right hand side
      d[is] = o_old[is-1] * ( - aux4 - aux5 ) + o_old[is] * ( -1 + 4 * aux4
 aux3 )
            + o_old[is+1] * ( - aux4 + aux5 ) + v * vega[is];
    }

    // DO: Initialize the boundary conditions:
    if( Y == CALL ){
      // Call boundary conditions
      b[0] = 1.0;
      c[0] = 0.0;
      d[0] = 0.0;

      a[ns] = 0.0;
      b[ns] = 1.0;
      d[ns] = ( s[ns] - K ) * exp( ( r - rd - rb ) * ( T - t[it] ));
    } else if( Y == PUT ){
      // Put boundary conditions
      b[0] = 1.0;
      c[0] = 0.0;
      d[0] = ( K - s[ns] ) * exp( - r * ( T - t[it] ));

      a[ns] = 0.0;
      b[ns] = 1.0;
      d[ns] = 0.0;
    }

    // DO: Solve the matrix for the new options values
    // DO: run the forward elimination first
    for( is = 1; is < ns; is++ ){
      aux6 = -  a[is] / b[is-1];
      b[is] += c[is-1] * aux6;
      d[is] += d[is-1] * aux6;
    }

    // DO: run backwards elimination to get the options values
```

Figure 6 (cont)

```
    o[ns] = d[ns];
    for( is = ns-1; is > 0; is-- ){
      o[is] = ( d[is] - c[is] * o[is+1] ) / b[is];
    }
    o[0] = d[0];

    // DO: compute the sensitivities
    for( is = 1; is < ns; is++ ){
      o[is] = max( 0.0, o[is] );  // option value should not be negative
      delta[is] = ( o[is+1] - o[is-1] ) / ( ds + ds );
      gamma[is] = min( MAX_GAMMA,
                max( 0.0, ( o[is+1] - 2 * o[is] + o[is-1] ) / ds2 ) );
      theta[is] = ( o_old[is] - o[is] ) / dt;
      vega[is]  = 0.0;  // in this software vega need not be computed
                        // because we assume v zero

      if( gamma[is] > 0.0 ){
    m[is] = max( 0.0, 1.0 + g - min( m1 * pow( gamma[is], 1.0/3.0 ),
                     2.0 * R * delta[is] * delta[is] / gamma[is] ) );
      } else {
    m[is] = 1.0 + g;
      }

      // DO: save the previous step for the options
      o_old[is] = o[is];
    }

    // DO: Save the computed values into the arrays
    for( is = 0; is <= ns; is++ ){
      O[it][is]     = o[is];
      Delta[it][is] = delta[is];
      Gamma[it][is] = gamma[is];
      Theta[it][is] = theta[is];
      Vega[it][is]  = vega[is];
      M[it][is]   = m[is];
    }
  }

  // INV: the values have been computed up to the current time

  // Return of the computed derivative's values and sensitivities
  // is done throughthe globally defined arrays O, Delta, Gamma, Theta, Vega

}
```

Figure 7

No mystery behind the smile

Milan Kratka presents a risk-adjusted pricing model that matches the market prices of options more closely

Classical option theory derives option values from the so-called arbitrage replication process: investors who buy options and hedge them continuously and without cost are expected to generate a risk-free return equal to that from Treasury bonds. These idealistic assumptions imply that the option model is not consistent with real markets. In real life, every investor rehedges at discrete time intervals, pays trading costs, is exposed to potential price gap and volatility risks, and experiences positive portfolio variance.

The pricing model we present here helps close this gap between theory and practice. Risk-adjusted pricing incorporates all the real-life factors into a simple but comprehensive model that is consistent with the market. It enhances the old "no-arbitrage" paradigm with the more realistic notion of value-at-risk. We will try to explain the main ideas behind this breakthrough methodology.

Standard assumptions

Suppose that an investor prices an option O on a stock S that pays zero dividends and follows the typical, constant-volatility lognormal process:

$$dS = \mu S dt + \sigma S d\tilde{z}$$

where μ denotes the drift (or the trend) and σ denotes the volatility. Let us assume we buy N options and short-sell δN stocks as a hedge. We plan to rehedge this portfolio by trading in stock. Let us denote the rehedging time interval by Δ_t. This should reflect the trading strategy chosen by the investor and, in general, will vary with the time to expiry, underlying price and other factors. Since no portfolio can be hedged continuously or without cost, the investor needs to specify a rehedging rule. For example, we might simply want to hedge in the most advantageous way. So we ask our quants to figure out how to do this. They may come up with strategies such as:

- ☐ rehedge on sunny days only;
- ☐ rehedge every hour;
- ☐ rehedge with every 1% change in the stock price;
- ☐ rehedge with every 0.05 change in delta; or
- ☐ leave rehedging strategy to the best trader.

What would be the optimal strategy? Rehedging too frequently accumulates high trading costs. Hedging too infrequently results in high portfolio variance. The risk-adjusted pricing methodology offers a natural solution to the problem of specifying an optimal rehedging strategy.

Risk-adjusted pricing

We have now set up a portfolio that will be rehedged in a time interval Δ_t. Over this period, we expect it to generate a return equal to that from risk-free securities, but it should also pay for the trading costs and compensate us for having positive portfolio variance and for taking volatility exposure risk. Though the transactions will occur at discrete times, we can locally estimate the rate of trading costs by spreading them over the time between the rehedging points, as is the customary practice with the dividend rate. We will estimate the rate of the portfolio variance similarly.

Transaction fees, trading strategies and risk aversion differ from one investor to another. An option may seem to be undervalued for one investor and overpriced for another. The market price of a liquid security reflects the prevailing expectations among all active investors. Who would not like to have a pricing model which is both in line with the market and reflects investor-specific conditions and expectations? The latter is not quite possible without the former. Risk-adjusted pricing allows us to model both.

Let us denote the risk-free rate by r, the stock borrowing rate by r_B, the expected rate of trading costs by r_{TC}, the expected rate of premium for taking positive variance risk by r_{VAR} and the expected rate of premium for volatility exposure risk by r_σ. The average change $E[dP]$ of the portfolio value in an infinitesimal time increment dt should then satisfy:

$$\begin{aligned} E[dP] &= E[dO] - E[\delta dS] \\ &= \left(\partial_t O + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} O\right)dt + E\left[(\partial_S O - \delta)dS\right] \\ &= r(O - \delta S)dt + r_B \delta S dt + \left(r_{VAR} + r_{TC} + r_\sigma\right)dt \end{aligned}$$

Rehedging to the market delta $\partial_S O$ would make the expected value $E[(\partial_S O - \delta)dS]$ equal zero, and the portfolio would have zero variance over the time dt. On the other hand, rehedging consistently to a different delta δ would make the expectation term equal to $(\partial_S O - \delta)\mu S dt$, and it would increase the expected portfolio variance by $(\partial_S O - \delta)^2\sigma^2 S^2 dt$. For investors with strong trend expectations, it is optimal to trade to a trend-adjusted delta that differs from the market delta.

In general, it is difficult to estimate the future trend term μ from the market. It is a subjective issue. If the trend was known and had significant value, the underlying price would have already been adjusted accordingly. There is much uncertainty about trends. Maybe that is why they are often referred to by the more moderate term "drifts". Each time we buy a stock, somebody sells it. Similarly, some investors might underhedge, while others might overhedge. For the market modelling we could simply assume that most traders hedge to the market delta. For investors with a strong opinion about the trend, or even for a market reflecting these opinions, rehedging to a trend-adjusted delta should be more advantageous, since it should increase expected return.

Portfolio variance risk premium

How much, in premium, should investors be compensated for positive portfolio variance risk? It is safe to say that the higher the variance, the higher the premium. As the variance of a normal process increases linearly in time, it may seem natural to expect the premium to follow the portfolio variance linearly. This is similar to portfolio performance rating using

Sharpe Ratios. If we denote the variance risk coefficient by R, the rate of the portfolio variance risk premium is:

$$r_{VAR} = \frac{R \cdot \mathrm{var}(\Delta P)}{\Delta_t}$$

You can also consider R as a risk aversion coefficient. The variance of the portfolio between two rehedging points can be estimated as:

$$\begin{aligned}\mathrm{var}(\Delta P) &= E\left[\left(\Delta P - E[\Delta P]\right)^2\right] \\ &= E\left[\frac{\left(\partial_S O(t+\Delta_t) - \delta(t)\right)^2}{(\Delta S)^2}(\Delta S)^4\right] \\ &= E\left[\Gamma^2(\Delta S)^4 + \left(\partial_S O(t) - \delta(t)\right)^2(\Delta S)^2\right] \\ &\simeq 3\Gamma^2\sigma^4 S^4 \Delta_t^2 + \left(\partial_S O - \delta\right)^2 \sigma^2 S^2 \Delta_t\end{aligned}$$

where ΔS denotes the change in the underlying price between rehedging times t and $t+\Delta_t$, and $\Gamma = \partial_{SS}O$ denotes market gamma, as is usual. The rate of the portfolio variance risk premium is then:

$$r_{VAR} = 3R\Gamma^2\sigma^4 S^4 \Delta_t + R\left(\partial_S O - \delta\right)^2 \sigma^2 S^2$$

Notably, the longer the time between rehedging Δ_t, the higher the rate of the variance risk premium r_{VAR}.

For a risk-averse investor with a strong opinion about the trend, the optimal delta to rehedge to is the one that maximises the risk-adjusted, trend-related rate:

$$\left(\partial_S O - \delta\right)\left(\mu - r + r_B\right)S - R\left(\partial_S O - \delta\right)^2 \sigma^2 S^2$$

That delta is:

$$\delta = \partial_S O - \frac{\mu - r + r_B}{2R\sigma^2 S}$$

This trend-adjusted delta makes the risk-adjusted, trend-related rate $(\mu - r + r_B)^2/4R\sigma^2$. A moderate up-trend would cause option holders to be hedged with a lower delta than the market delta. Trends seem to have a relatively high impact on options that are not at-the-money. Whatever the direction and strength of future trends, their presence and market anticipation will help to enlarge the effect of higher implied volatility at the tails.

Trading costs estimator

An investor's trading costs typically depend on the number of contracts traded, type of order, size of bid-ask spread, bid and ask volumes, exchange fees and a transaction fees structure negotiated with a broker. Let us denote an aggregate trading cost per contract, when the size of a trade is n contracts, by C(n). It is typically a non-increasing function of a trade size, $C'(n) \leq 0$. The larger the trading size, the higher the total commissions but the lower the total cost per traded share. It does not need to be positive, as heavy buying may move the market above the average cost of the trade, causing an immediate paper gain. For market modelling, we may assume that the prevailing aggregate trading cost per contract is constant $C(n) = C$. Investors should specify their own trading costs structure. It will affect investors' expected option theoretical values and their choice of rehedging strategy. Trading costs alone would have a relatively small price effect but combined with discrete hedging, positive variance risk and volatility exposure premiums can have a significant impact.

The expected rate of trading costs:

$$r_{TC} = \frac{C\left(N|\Delta\delta|\right)|\Delta\delta|}{\Delta_t}$$

reflects that we will trade $N|\Delta\delta|$ shares and spread out the cost over a time interval Δ_t. The average amount of shares traded per option in each delta rehedge adjustment can be estimated using Itô's lemma:

$$\begin{aligned}E\left[|\Delta\delta|\right] &= E\left[\left|\partial_t\delta\Delta_t + \partial_S\delta\Delta S + \tfrac{1}{2}\partial_{SS}\delta(\Delta S)^2\right|\right] \\ &\simeq \sigma S|\partial_S\delta|E\left[|d\tilde{z}|\right] \\ &= \sigma S|\Gamma|\sqrt{\tfrac{2}{\pi}}\sqrt{\Delta_t}\end{aligned}$$

If the time between rehedging was long enough, the expected value could be calculated more precisely, and would marginally depend on the trend μ. The rate of trading costs can finally be expressed as:

$$r_{TC} = C\left(N\sqrt{\tfrac{2}{\pi}}\sigma S|\Gamma|\sqrt{\Delta_t}\right)\sqrt{\tfrac{2}{\pi}}\sigma S|\Gamma|\frac{1}{\sqrt{\Delta_t}}$$

Notably, the longer the time between rehedging Δ_t, the lower the trading costs rate r_{TC}.

The higher the volatility sensitivity of the portfolio (which equals the option's vega), the higher the volatility risk we are exposed to. How should we model the volatility risk exposure? It will depend on our anticipation of volatility change, which is a subject for a separate project, a volatility model. We can model the volatility risk premium by $r_\sigma = v|\partial_\sigma O|$, say, reflecting our portfolio sensitivity to any change in underlying volatility. The higher the vega, the higher the premium. The specific form of the volatility risk premium coefficient should come from our volatility model. It should reflect the probability of the volatility change as well as the size of its probable change.

Optimal hedging strategies

If we believe in the efficiency of financial markets, we might suppose that market option prices reflect the market value of risk, average prevailing trading costs and competitive rehedging strategies. There is never a guarantee that the free market will value options in any particular way. But we can assume it does, build a pricing model and use it to verify our presumption.

Given investor-specific trading costs C(n) and an investor-specific variance risk coefficient R, the optimal rehedging strategy is identified by minimising the sum of r_{VAR} and r_{TC}, the variance premium and the trading cost terms:

$$r_{VAR} + r_{TC} = 3R\sigma^4 S^4 \Gamma^2 \Delta_t + C\left(\sqrt{\tfrac{2}{\pi}}N\sigma S|\Gamma|\sqrt{\Delta_t}\right)\sqrt{\tfrac{2}{\pi}}\sigma S|\Gamma|\frac{1}{\sqrt{\Delta_t}}$$

where the rehedging time Δ_t varies in a reasonable range. This may span a fraction of an hour to a fraction of a year. The gamma, Γ, should be equal to the market gamma $\partial_S O$. The simple task of finding the minimum can be done approximately or even precisely. There is no need to write elaborate papers about what an optimal hedging strategy in the presence of trading costs ought to be. If the trading costs per share, C, do not depend on trade size, the optimal rehedging time would be:

$$\Delta_t = \frac{k^2}{\sigma^2 S^2 \Gamma^{2/3}}$$

where we have denoted the constant:

$$k = \left(C / 3R\sqrt{2\pi}\right)^{1/3}$$

This translates to a very simple strategy of rehedging with every stock change of:

$$\Delta S = \pm\sigma S\sqrt{\Delta_t} = \pm k\Gamma^{-1/3}$$

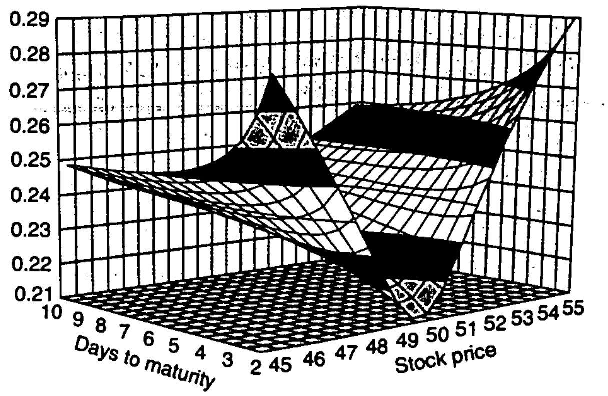
This is probably close to a rehedging rule you have already used. If not, feel free to upgrade now!

In line with our intuition, we shall trade more often with higher risk aversion R, lower trading costs C or higher gamma Γ. In the limiting case of zero transaction costs, we would trade continuously, as assumed by the Black-Scholes model.

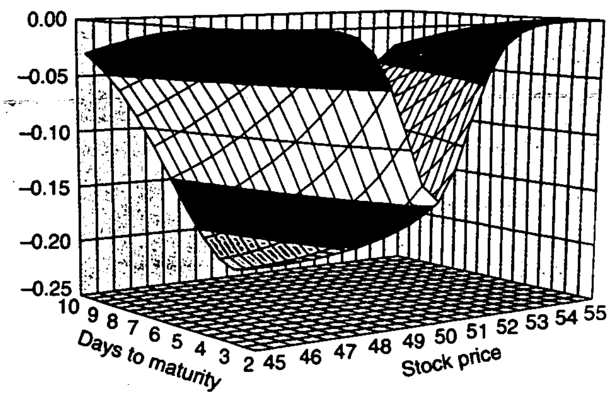
The assumption of competitive hedging in the market will then yield the sum of the variance premium and the trading costs terms to be:

$$r_{VAR} = r_{TC} = \tfrac{1}{2}m\sigma^2 S^2 \Gamma\Gamma^{1/3}$$

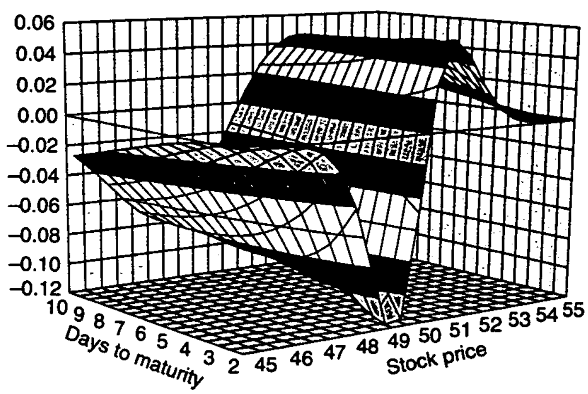
1. Black-Scholes implied volatility



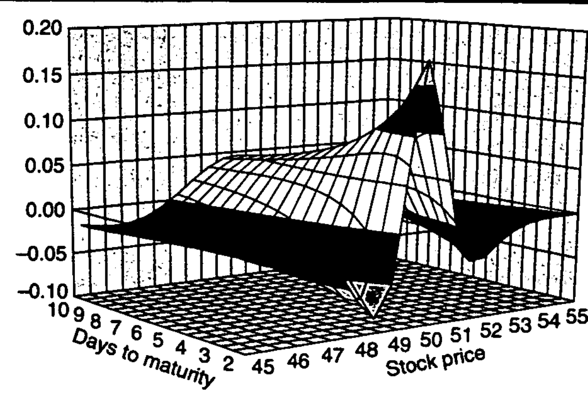
2. Risk-adjusted call price minus Black-Scholes call price



3. Risk-adjusted delta minus Black-Scholes delta



4. Risk-adjusted gamma minus Black-Scholes gamma



where we have denoted the constant $m = (324RC^2/\pi)^{1/3}$. This was the last piece needed to finalise a surprisingly simple risk-adjusted pricing equation:

$$\partial_t O + \frac{1}{2}\sigma^2 S^2\left(1 - m\Gamma^{1/3}\right)\Gamma = r(O - \delta S) + r_B \delta S$$

It can be made more complete by including the drift term $-(\mu - r + r_B)^2/4R\sigma^2$ and the volatility risk premium term that in our simplified form was $v|\partial_\sigma O|$. It can be solved numerically backwards in time by finite difference scheme methods. The three-dimensional sample plots above (figures 1–4) have been produced by choosing an underlying volatility of 30%, a risk parameter of 0.9 and an interest rate of 6%. For the purposes of this paper, we have left out the technical details of the model implementation.[1] We will just note that the rate of the portfolio variance should not be greater than the variance $\delta^2\sigma^2 S^2$ of an unhedged option position, which caps the variance risk premium that would have otherwise spiked up at the strike price at the option's maturity.

Volatility smile

There are just two market parameters in the simple risk-adjusted option pricing equation – the anticipated volatility of the underlying price, σ, and the so-called risk parameter m – although we would consider the volatility risk premium coefficient v, trend parameter μ and risk aversion coefficient R for a more detailed option model. The volatility risk premium coefficient measures our exposure to the risk that the market could change its mind about the anticipated volatility. Its effect on option pricing should diminish with time to maturity, and thus it would not explain sharpening of the so-called volatility smile. The risk parameter is the one responsible for its sharpening, as its effect on option pricing magnifies with gamma, as if the options with higher gamma were priced with lower volatility. The trendiness of the market μ helps to sharpen the smile at the tails.

In any case, it is quite nonsensical to assume a different underlying volatility for different options. There can be only one anticipated underlying volatility, no matter which options we look at. Even if we could incorporate fat tails, price gaps, complex supply-demand relations or even autocorrelation, there would still be only one underlying process with only one volatility. The notion of implied volatilities points out how far away the market pricing is from an idealised risk-neutral Black-Scholes world.

So, why do we observe volatility smiles? Because there are costs and risks associated with options trading, as we have seen and become capable of modelling. There is no mystery behind the smile any more. The risk-adjusted pricing methodology may not replicate the market pricing perfectly, but it does explain quite a significant part of its stable dynamics. It provides an edge in understanding option valuation and optimal hedging strategies. It shows how to incorporate various market risks quantitatively into an option pricing equation. It will even save resources that would otherwise be spent on artificial implied volatility surfaces modelling. Finally, it is simple and easy to put into production.

We have chosen to model market risks in a simplified way to demonstrate the natural power of the risk-adjusted pricing methodology. It is just a first step in pointing out the new dimensions possible in the financial modelling of a not quite risk-neutral world. ■

[1] *Feel free to contact the author if you have any questions*

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